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MAXIMUM LIKELIHOOD ESTIMATION FOR AN AUTOREGRESSIVE PROCESS WIT--ETC(U)

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⑥ Maximum Likelihood Estimation For An Autoregressive  
Process With Missing Observations

by

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Abstract

Three methods are proposed for estimation of the parameters of an autoregressive process of order  $p$  with missing observations. These methods are based on the maximum likelihood approach and use the EM algorithm, the Newton-Raphson method and the method of scoring, which are applied to the likelihood equations. Finally, comparison on those methods is also discussed.

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## 1. Introduction

An autoregressive process  $\{y_t, t = 0, \pm 1, \dots\}$  of order  $p$  is defined by

$$(1.1) \quad \sum_{i=0}^p \gamma_i y_{t-i} = \epsilon_t, \quad t = 0, \pm 1, \dots,$$

where  $\gamma_0 = 1$  and  $\{\epsilon_t\}$  is a sequence of uncorrelated random variables with mean 0 and common variance  $\sigma^2$ . We assume that the roots of  $\sum_{i=0}^p \gamma_i z^i = 0$  are outside

the unit disc. The process (1.1) is completely specified by

$\phi \equiv (\gamma_1, \dots, \gamma_p, \sigma^2)'$  when the  $\epsilon_t$  are assumed to be normally distributed.

Throughout this paper we shall assume normality of  $\epsilon_t$ .

Usually statistical inference is based on a set of  $T$  consecutive observations on  $y_t$ . Let

$$(1.2) \quad \underline{y} = (y_1, y_2, \dots, y_T)'$$

and let  $P$  be a permutation matrix such that  $P\underline{y} = (\underline{s}', \underline{m}')'$ , where  $\underline{s}$  is a  $(T-m) \times 1$  vector and  $\underline{m}$  is an  $m \times 1$  vector, with the ordering in  $\underline{s}$  and  $\underline{m}$  preserved. Suppose only observations in  $\underline{s}$  are available and those in  $\underline{m}$  are missing. Our goal here is to obtain maximum likelihood estimates of  $\phi$ .

For any  $T \times T$  matrix  $C$ , let us define  $\underline{C}_{ss}$ ,  $\underline{C}_{sm}$ ,  $\underline{C}_{ms}$  and  $\underline{C}_{mm}$  to be the  $(T-m) \times (T-m)$ ,  $(T-m) \times m$ ,  $m \times (T-m)$  and  $m \times m$  matrices, respectively, satisfying

$$(1.3) \quad \underline{P} \underline{C} \underline{P}' = \begin{pmatrix} \underline{C}_{ss} & \underline{C}_{sm} \\ \underline{C}_{ms} & \underline{C}_{mm} \end{pmatrix}.$$

For the rest of this paper, let  $f(\underline{y}|\phi)$  denote the probability density function of  $\underline{y}$ ,  $f(\underline{s}|\phi)$  denote the probability density function of  $\underline{s}$ ,  $f(\underline{m}|\underline{s}, \phi)$  denote the conditional probability density function of  $\underline{m}$  given  $\underline{s}$ ,  $\log f(\underline{y}|\phi)$  denote the log

likelihood function based on  $\underline{y}$  and  $\log f(\underline{s}|\underline{\phi})$  denote the log likelihood function based on  $\underline{s}$ . We assume that the maximum likelihood solutions satisfy

$$(1.4) \quad \frac{\partial \log f(\underline{s}|\underline{\phi})}{\partial \underline{\phi}} = \underline{0}.$$

## 2. Some basic results

Assume that  $\underline{y}$  is distributed as multivariate normal with mean  $\underline{0}$  and covariance matrix  $\underline{\Sigma}$ , that is,  $f(\underline{y}|\underline{\phi})$  is given by

$$(2.1) \quad f(\underline{y}|\underline{\phi}) = \frac{1}{\sqrt{(2\pi)^T |\underline{\Sigma}|}} \exp \left\{ -\frac{1}{2} \underline{y}' \underline{\Sigma}^{-1} \underline{y} \right\}.$$

Then  $\underline{P}\underline{y} = (\underline{s}', \underline{m}')'$  is distributed as multivariate normal with mean  $\underline{0}$  and covariance matrix  $\underline{P} \underline{\Sigma} \underline{P}'$ . Since  $\underline{P} \underline{P}' = \underline{I}_T$ , where  $\underline{I}_T$  is the  $T \times T$  identity matrix,

$$(2.2) \quad \begin{aligned} (\underline{P} \underline{\Sigma} \underline{P}')^{-1} &= \underline{P} \underline{\Sigma}^{-1} \underline{P}' \\ &= \frac{1}{\sigma^2} \underline{P} \underline{M} \underline{P}' \\ &= \frac{1}{\sigma^2} \begin{pmatrix} \underline{M}_{ss} & \underline{M}_{sm} \\ \underline{M}_{ms} & \underline{M}_{mm} \end{pmatrix}, \end{aligned}$$

where  $\sigma^2 \underline{\Sigma}^{-1} = \underline{M}$ , and  $\underline{M}_{ss}$ ,  $\underline{M}_{sm}$ ,  $\underline{M}_{ms}$  and  $\underline{M}_{mm}$  are as defined by (1.3). Also, by (1.3), we get

$$(2.3) \quad \underline{\underline{P}} \underline{\underline{\Sigma}} \underline{\underline{P}}' = \begin{pmatrix} \underline{\underline{\Sigma}}_{ss} & \underline{\underline{\Sigma}}_{sm} \\ \underline{\underline{\Sigma}}_{ms} & \underline{\underline{\Sigma}}_{mm} \end{pmatrix}.$$

Therefore from (2.2) and (2.3), it follows that

$$(2.4) \quad [\text{Cov}(\underline{s}|\underline{\phi})]^{-1} = \underline{\underline{\Sigma}}_{ss}^{-1} = \frac{1}{\sigma^2} [\underline{\underline{M}}_{ss} - \underline{\underline{M}}_{sm} \underline{\underline{M}}_{mm}^{-1} \underline{\underline{M}}_{ms}],$$

$$(2.5) \quad [\text{Cov}(\underline{m}|\underline{s}, \underline{\phi})]^{-1} = [\underline{\underline{\Sigma}}_{mm} - \underline{\underline{\Sigma}}_{ms} \underline{\underline{\Sigma}}_{ss}^{-1} \underline{\underline{\Sigma}}_{sm}]^{-1} \\ = \frac{1}{\sigma^2} \underline{\underline{M}}_{mm},$$

$$(2.6) \quad E[\underline{m}|\underline{s}, \underline{\phi}] = \underline{\underline{\Sigma}}_{ms} \underline{\underline{\Sigma}}_{ss}^{-1} \underline{s} \\ = - \underline{\underline{M}}_{mm}^{-1} \underline{\underline{M}}_{ms} \underline{s},$$

and

$$(2.7) \quad |\underline{\underline{M}}| = |\underline{\underline{P}} \underline{\underline{M}} \underline{\underline{P}}'| \\ = |\underline{\underline{M}}_{mm}| |\underline{\underline{M}}_{ss} - \underline{\underline{M}}_{sm} \underline{\underline{M}}_{mm}^{-1} \underline{\underline{M}}_{ms}|.$$

From (2.4) and (2.7), we obtain

$$(2.8) \quad f(\underline{s}|\underline{\phi}) = \left( \frac{1}{\sqrt{2\pi} \sigma^2} \right)^{T-m} \left( \frac{|\underline{\underline{M}}|}{|\underline{\underline{M}}_{mm}|} \right)^{\frac{1}{2}} \\ \exp \left\{ - \frac{1}{2\sigma^2} \underline{s}' [\underline{\underline{M}}_{ss} - \underline{\underline{M}}_{sm} \underline{\underline{M}}_{mm}^{-1} \underline{\underline{M}}_{ms}] \underline{s} \right\}.$$

Expressions (2.5) and (2.6) will be used in the following sections. Though (2.8) gives the expression for the probability density function of  $\underline{s}$ , we will not use

it to obtain the score function  $\frac{\partial \log f(\underline{s}|\underline{\phi})}{\partial \underline{\phi}}$ , due to the simplicity of  $\frac{\partial \log f(\underline{y}|\underline{\phi})}{\partial \underline{\phi}}$  and Lemma 1 in section 3. We will use (2.8) in proving the asymptotic properties of the estimates in a subsequent paper. Under suitable conditions, the estimates of  $\underline{\phi}$  based on the Newton-Raphson method and the method of scoring are shown to be  $\sqrt{T-m}$  - consistent, asymptotically normal and one-step asymptotically efficient if the initial estimates are  $\sqrt{T-m}$  - consistent.

### 3. Estimation

Let

$$(3.1) \quad \underline{y} = (y_1, \dots, y_p)'$$

and

$$(3.2) \quad \underline{y}_\mu = (1, \underline{y}')'$$

Then (see Anderson (1971), sec 6.2, and Box and Jenkins (1976), sec 7.A.5)

$$(3.3) \quad \begin{aligned} \log f(\underline{y}|\underline{\phi}) &= -\frac{T}{2} \log (2\pi \sigma^2) + \frac{1}{2} \log |\underline{M}| - \frac{1}{2\sigma^2} \underline{y}' \underline{M} \underline{y} \\ &= -\frac{T}{2} \log (2\pi \sigma^2) + \frac{1}{2} \log |\underline{M}| - \frac{1}{2\sigma^2} \underline{y}_\mu' \underline{D} \underline{y}_\mu, \end{aligned}$$

where the elements  $m_{st}$  of the  $T \times T$  matrix  $\underline{M}$  is given by

$$(3.4) \quad \begin{aligned} m_{st} &= m_{T+1-t, T+1-s} \\ &= \sum_{j=0}^{\min(s,t)-1} y_j y_{j+|s-t|}, & s, t = 1, \dots, p, \\ &= \sum_{j=0}^{p-|s-t|} y_j y_{j+|s-t|}, & \max(s,t) \geq p+1, \\ &= 0, & \min(s,t) \leq T-p, \\ & & |s-t| = 0, 1, \dots, p, \\ & & |s-t| = p+1, \dots, \end{aligned}$$

and the element  $d_{ij}$  of the  $(p+1) \times (p+1)$  matrix  $\underline{D}$  is given by

$$(3.5) \quad d_{ij} = \underline{y}' \underline{A}_{ij} \underline{y}$$

with the element  $a_{mn}$  of the  $T \times T$  matrix  $\underline{A}_{ij}$  given

by

$$(3.6) \quad a_{mn} = 1, \quad \text{if } (m,n) = (i+s, j+s),$$

$$s = 0, 1, \dots, T+1 - (i+j),$$

$$= 0, \quad \text{otherwise.}$$

From (3.3) we obtain

$$(3.7) \quad \frac{\partial \log f(\underline{y}|\underline{\phi})}{\partial \gamma_j} = \frac{1}{2} \frac{\partial \log |\underline{M}|}{\partial \gamma_j} - \frac{1}{\sigma^2} \sum_{i=0}^p \gamma_i d_{i+1,j+1},$$

$$j = 1, \dots, p,$$

and

$$(3.8) \quad \frac{\partial \log f(\underline{y}|\underline{\phi})}{\partial \sigma^2} = -\frac{T}{2\sigma^2} + \frac{1}{2\sigma^4} \underline{\gamma}' \underline{D} \underline{\gamma}.$$

It is  $\frac{\partial \log f(\underline{s}|\underline{\phi})}{\partial \underline{\phi}}$  that is of interest to us and not  $\frac{\partial \log f(\underline{y}|\underline{\phi})}{\partial \underline{\phi}}$ , since observations in  $\underline{m}$  are missing. However,  $\frac{\partial \log f(\underline{s}|\underline{\phi})}{\partial \underline{\phi}}$  can be derived from

$\frac{\partial \log f(\underline{y}|\underline{\phi})}{\partial \underline{\phi}}$  as indicated in the following lemma.

**LEMMA 1.** (Orchard and Woodbury (1972)).

$$(3.9) \quad \frac{\partial \log f(\underline{s}|\underline{\phi})}{\partial \underline{\phi}} = E \left[ \frac{\partial \log f(\underline{y}|\underline{\phi})}{\partial \underline{\phi}} \mid \underline{s}, \underline{\phi} \right].$$

Proof. The result follows immediately from

$$(3.10) \quad E \left[ \frac{\partial \log f(\underline{m} | \underline{s}, \underline{\phi})}{\partial \underline{\phi}} \mid \underline{s}, \underline{\phi} \right] = \underline{0}.$$

It is clear from (3.7) - (3.9) that

$$(3.11) \quad \frac{\partial \log f(\underline{s} | \underline{\phi})}{\partial \gamma_j} = \frac{1}{2} \frac{\partial \log |\underline{M}|}{\partial \gamma_j} - \frac{1}{\sigma^2} \sum_{i=0}^p \gamma_i E \left[ d_{i+1,j+1} \mid \underline{s}, \underline{\phi} \right]$$

$$j = 1, \dots, p,$$

and

$$(3.12) \quad \frac{\partial \log f(\underline{s} | \underline{\phi})}{\partial \sigma^2} = - \frac{T}{2\sigma^2} + \frac{1}{2\sigma^4} \underline{\gamma}'_{\mu} E \left[ \underline{D} \mid \underline{s}, \underline{\phi} \right] \underline{\gamma}_{\mu}.$$

The term  $\log |\underline{M}|$  is  $O(1)$  (see Hannan (1973), e.g.) while  $d_{ij}$  is  $O_p(T)$ . The effect of neglecting  $\log |\underline{M}|$  is negligible for moderate or large  $T$ , and we shall neglect  $\log |\underline{M}|$  and other negligible terms henceforth. From (3.11) and (3.12), it follows that the likelihood equations are given by

$$(3.13) \quad \sum_{g=0}^p \gamma_g E \left[ d_{g+1,j+1} \mid \underline{s}, \underline{\phi} \right] = 0, \quad j = 1, \dots, p,$$

and

$$(3.14) \quad \sigma^2 = \frac{1}{T} \underline{\gamma}'_{\mu} E \left[ \underline{D} \mid \underline{s}, \underline{\phi} \right] \underline{\gamma}_{\mu}.$$

When there are no missing observations,  $E \left[ d_{g+1,j+1} \mid \underline{s}, \underline{\phi} \right] = d_{g+1,j+1}$  does not involve unknown parameters. Then the equations are linear in  $\gamma_i$ ,  $i = 1, \dots, p$ , and are the Yule-Walker equations. When missing observations do occur,

$E \left[ d_{g+1,j+1} \mid \underline{s}, \underline{\phi} \right]$  involves unknown parameters and (3.13) and (3.14) are highly non-linear in the unknown parameters. In fact, from (2.5) and (2.6),

$$(3.15) \quad E \left[ d_{g+1,j+1} \mid \underline{s}, \underline{\phi} \right]$$

$$= \underline{s}' \left[ (\underline{A}_{g+1,j+1})_{ss} - (\underline{A}_{g+1,j+1})_{sm} \underline{K} \right.$$

$$\left. - \underline{K}' (\underline{A}_{g+1,j+1})_{ms} + \underline{K}' (\underline{A}_{g+1,j+1})_{mm} \underline{K} \right] \underline{s}$$

$$+ \sigma^2 \text{tr} (\underline{A}_{g+1,j+1}) - 1.$$



where  $\underline{K} = \underline{M}_{mm}^{-1} \underline{M}_{ms}$ , and the matrices involved are as defined in (1.3), (3.4) and (3.6). Therefore solutions of (3.13) and (3.14) are not straightforward and iterative procedures have to be used.

We propose the following three methods of solving (3.13) and (3.14): the EM algorithm, the Newton-Raphson method and the method of scoring.

a. The EM algorithm. Since  $\underline{\phi}$  is to be estimated, it is natural that one replace  $E[\cdot | \underline{s}, \underline{\phi}]$  in (3.13) and (3.14) by  $E[\cdot | \underline{s}, \underline{\phi}_i]$ , where  $\underline{\phi}_i$  is some estimated value of  $\underline{\phi}$ , and obtain  $\underline{\phi}_{i+1}$  iteratively by solving

$$(3.16) \quad \sum_{g=0}^p (\gamma_g)_{i+1} E[d_{g+1,j+1} | \underline{s}, \underline{\phi}_i] = 0.$$

$$j = 1, \dots, p,$$

and

$$(3.17) \quad \sigma_{i+1}^2 = \frac{1}{T} (\underline{\gamma}_\mu)_i' E[\underline{D} | \underline{s}, \underline{\phi}_i] (\underline{\gamma}_\mu)_i.$$

Here  $(\gamma_g)_j$  and  $(\gamma_\mu)_j$  denote the estimates of  $\gamma_g$  and  $\gamma_\mu$ , respectively, at the  $j$ -th iteration. As shown in Tan (1979), the above method gives the same solutions as the EM algorithm proposed by Dempster, Laird and Rubin (1977).

b. The Newton-Raphson Method. From (3.11) and (3.12), we obtain

$$(3.18) \quad \theta_{kj} \equiv - \frac{\partial^2 \log f(\underline{s} | \underline{\phi})}{\partial \gamma_k \partial \gamma_j} = \frac{1}{\sigma^2} \left\{ \sum_{g=0}^p \gamma_g \frac{\partial}{\partial \gamma_k} E[d_{g+1,j+1} | \underline{s}, \underline{\phi}] \right.$$

$$\left. + E[d_{k+1,j+1} | \underline{s}, \underline{\phi}] \right\},$$

$$j, k = 1, \dots, p,$$

$$(3.19) \quad \theta_{j,p+1} \equiv - \frac{\partial^2 \log f(\underline{s}|\underline{\phi})}{\partial \gamma_j \partial \sigma^2} = \sum_{g=0}^p \gamma_g \left\{ \frac{1}{\sigma^2} \text{tr} (A_{g+1,j+1})_{mm} M_{mm}^{-1} \right. \\ \left. - \frac{1}{\sigma^4} E[d_{g+1,j+1} | \underline{s}, \underline{\phi}] \right\},$$

$$j = 1, \dots, p,$$

and

$$(3.20) \quad \theta_{p+1,p+1} \equiv - \frac{\partial^2 \log f(\underline{s}|\underline{\phi})}{\partial (\sigma^2)^2} = - \left[ \frac{T}{2\sigma^4} + \frac{1}{2} \gamma_\mu' B \gamma_\mu \right],$$

where the element  $b_{ij}$  of the  $(p+1) \times (p+1)$  matrix  $B$  is given by

$$(3.21) \quad b_{ij} = - \frac{2}{\sigma^6} E[d_{ij} | \underline{s}, \underline{\phi}] + \frac{1}{\sigma^4} \text{tr} (A_{i+1,j+1})_{mm} M_{mm}^{-1}.$$

Thus, the Newton-Raphson method leads to the following set of equations:

$$(3.22) \quad \underline{\phi}_{i+1} - \underline{\phi}_i = \frac{\partial \log f(\underline{s}|\underline{\phi})}{\partial \underline{\phi}} \bigg|_{\underline{\phi}_i},$$

where the element  $\theta_{ij}$  of  $\underline{\theta}$  is as given by (3.18) - (3.21) and  $\frac{\partial \log f(\underline{s}|\underline{\phi})}{\partial \underline{\phi}}$

is as given by (3.11) (without the first term on the right-hand side) and (3.12).

c. The Method of Scoring. From (3.15), (3.18) - (3.21), we obtain

$$(3.23) \quad \phi_{ij} \equiv E[\theta_{ij}] = \frac{1}{\sigma^2} \{T \sigma(i-j) + \sum_{g=0}^p \gamma_g E \frac{\partial}{\partial \gamma_i} E[d_{g+1,j+1} | \underline{s}, \underline{\phi}]\},$$

$$i, j = 1, \dots, p,$$

$$(3.24) \quad \phi_{j,p+1} \equiv E[\theta_{j,p+1}] = \frac{1}{\sigma^2} \sum_{g=0}^p \gamma_g \text{tr} (A_{g+1,j+1})_{mm} M_{mm}^{-1},$$

$$j = 1, \dots, p,$$

and

$$(3.25) \quad \phi_{p+1,p+1} \equiv E[\theta_{p+1,p+1}] = -\frac{T}{2\sigma^4} + \frac{1}{\sigma^6} T \sum_{i,j=0}^p \gamma_i \gamma_j \sigma(i-j) \\ - \frac{1}{2\sigma^4} \sum_{i,j=0}^p \gamma_i \gamma_j \operatorname{tr} (A_{i+1,j+1})_{mm} M_{mm}^{-1},$$

where  $\sigma(k) = E[y_{t+k} y_t]$ . Thus, the method of scoring leads to the following set of equations:

$$(3.26) \quad \left. \frac{\partial \log f(\underline{s}|\underline{\phi})}{\partial \underline{\phi}} \right|_{\underline{\phi}_1} (\underline{\phi}_{i+1} - \underline{\phi}_1) = \left. \frac{\partial \log f(\underline{s}|\underline{\phi})}{\partial \underline{\phi}} \right|_{\underline{\phi}_1},$$

where the elements  $\phi_{ij}$  of  $\underline{\phi}$  are given by (3.23) - (3.25), and  $\left. \frac{\partial \log f(\underline{s}|\underline{\phi})}{\partial \underline{\phi}} \right|_{\underline{\phi}_1}$  is given by (3.11) (without the first term on the right-hand side) and (3.12).

We have used the fact that  $E[d_{i+1,j+1}] = [T - (i+j)] \sigma(i-j)$ , which can be approximated by  $T \sigma(i-j)$  for moderate or large  $T$ .

#### 4. Comparison of the methods of estimation

The estimates of  $\underline{\phi}$  based on the Newton-Raphson method, the method of scoring and the EM algorithm can be expressed in the following form

$$\left. \frac{\partial \log f(\underline{s}|\underline{\phi})}{\partial \underline{\phi}} \right|_{\underline{\phi}_1} (\underline{\phi}_{i+1} - \underline{\phi}_1) = \left. \frac{\partial \log f(\underline{s}|\underline{\phi})}{\partial \underline{\phi}} \right|_{\underline{\phi}_1},$$

where  $\left. \frac{\partial \log f(\underline{s}|\underline{\phi})}{\partial \underline{\phi}} \right|_{\underline{\phi}_1}$  is given by (3.11) (without the first term on the right-hand side) and (3.12). In the Newton-Raphson method, we have

$$\underline{H} = \underline{\Theta},$$

where  $\underline{\Theta}$  is given by (3.18) - (3.21). In the method of scoring, we have

$$\underline{\underline{H}} = \underline{\underline{\phi}},$$

where  $\underline{\underline{\phi}}$  is given by (3.23) - (3.25). In the EM algorithm, we have

$$\underline{\underline{H}} = \underline{\underline{\omega}},$$

where elements  $\omega_{ij}$  of  $\underline{\underline{\omega}}$  are given by

$$\begin{aligned} \omega_{gj} &= \omega_{jg} \\ &= \frac{1}{\sigma^2} E[d_{g+1,j+1} | \underline{\underline{s}}, \underline{\underline{\phi}}], & g, j = 1, \dots, p, \\ &= 0, & g = 1, \dots, p, j = p+1 \\ &= \frac{T}{2\sigma^4}, & g = p+1, j = p+1. \end{aligned}$$

Let

$$\underline{\underline{A}} = \underline{\underline{\Theta}} - \underline{\underline{\phi}},$$

$$\underline{\underline{B}} = \underline{\underline{\omega}} - \underline{\underline{\Theta}},$$

$$\underline{\underline{C}} = \underline{\underline{\omega}} - \underline{\underline{\phi}},$$

and

$$\begin{aligned} I_{\underline{\underline{m}} \underline{\underline{m}}}(\underline{\underline{\phi}}) &= E \left[ - \frac{\partial^2 \log f(\underline{\underline{y}} | \underline{\underline{\phi}})}{\partial \underline{\underline{\phi}} \partial \underline{\underline{\phi}}'} \right] \\ &= E \left[ - \frac{\partial^2 \log f(\underline{\underline{s}} | \underline{\underline{\phi}})}{\partial \underline{\underline{\phi}} \partial \underline{\underline{\phi}}'} \right] \\ &= E \left[ - \frac{\partial^2 \log f(\underline{\underline{y}} | \underline{\underline{\phi}})}{\partial \underline{\underline{\phi}} \partial \underline{\underline{\phi}}'} \right] - E[\underline{\underline{\Theta}}]. \end{aligned}$$

$I_m(\phi)$  is referred to as the loss information matrix in Orchard and Woodbury (1972). It follows that

$$E(A) = 0,$$

and also

$$\lim_{T-m \rightarrow \infty} \frac{E(B)}{T-m} = \lim_{T-m \rightarrow \infty} \frac{E(C)}{T-m} = \lim_{T-m \rightarrow \infty} \frac{I_m(\phi)}{T-m},$$

since  $\lim_{T-m \rightarrow \infty} \frac{E(\omega)}{T-m} = \lim_{T-m \rightarrow \infty} -E \frac{1}{T-m} \frac{\partial^2 \log f(\omega|\phi)}{\partial \phi \partial \phi'}$  (see Box and Jenkins (1976)

section 7.A.5). In general,  $\lim_{T-m \rightarrow \infty} \frac{I_m(\phi)}{T-m}$  is not negligible. For example, when  $p = 1$  in (1.1) and the process  $\{y_t\}$  is periodically observed for  $\alpha$  time points and then not observed for two time points, it can be shown that (see Tan (1979))

$$\lim_{T-m \rightarrow \infty} \frac{I_m(\phi)}{T-m} = \frac{1}{\alpha \sigma^2} \begin{pmatrix} \frac{\sigma^2(1+2\gamma_1^2)}{\lambda^2} (3 + \gamma_1^2 - \gamma_1^4) & -\frac{\gamma_1(1+2\gamma_1^2)}{\lambda} \\ -\frac{\gamma_1(1+2\gamma_1^2)}{\lambda} & \frac{1}{\sigma^2} \end{pmatrix}$$

where  $\lambda = (1 + \gamma_1^2 + \gamma_1^4)$ . It is easy to see that the above matrix is positive definite.

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